Simulation and Control of 2-Dimensional Anisotropic Heat Conduction

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Abstract. In this contribution, we provide a simulation and control approach for a two-dimensional heat conduction problem.
In particular, we spatially approximate the two-dimensional heat conduction problem with anisotropic thermal conductivity and transfer it via eigenvalue computation to a sampled-time state space model.
Furthermore, we design a static feed-forward filter to reach a reference temperature and a full state feedback with linear-quadratic regulation to guarantee a stable closed-loop behavior.
Finally, we demonstrate the per-formance of our closed-loop heat conduction system.

Introduction

Many industries, like semiconductor fabrication, require challenging thermal processing which need to be simulated and controlled very precisely, see e.g. [1]. The thermal dynamics modeling and its control design cover a wide range in the literature, depending on the geometry and research focus, see e.g. [1, 4] and [2, Ch. 2, 6, 8, 9]. The control design for small models can be simple and practical but they are limited for enhancements. In contrast, purely theoretical approaches might be too complex for realistic applications. In our contribution, we propose an extendable and easy-to-implement approach as a 2-dimensional geometry with multiple actuators along one boundary side and multiple sensors on the opposite. This is a simplified model of the realistic 3-dim. situation, see [3]. We approximate the spatial derivatives via finite differences and we obtain a high-dim. state space, which is solvable through eigenvalue computation in Section 1.

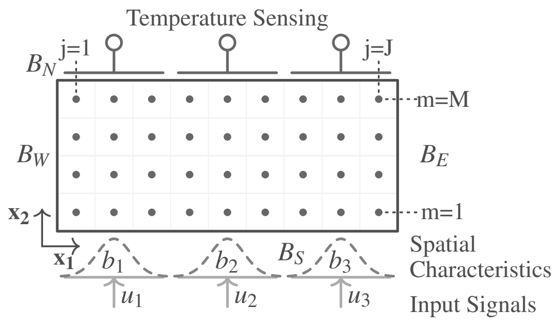

Figure 1: Geometry with finite difference nodes $x_{j,m}$, actuation on B_S and measurement on B_N.

Based on these results, we derive the time discrete solution, the state feedback with linear-quadratic regulation and the reference tracking in Section 2. Finally, we visualize in Section 3 the proper operating closed-loop behavior.

1 Two-dimensional Heat Conduction

We assume a rectangle $\Omega = (0,L)\times(0,W)$ with length $L>0$, width $W>0$, see Fig. 1. We note the position $x = (x_1,x_2)^\top \in \overline{\Omega}$. The rectangle has the boundary $\partial\Omega = \overline{\Omega}\setminus\Omega = B_W \cup B_E \cup B_S \cup B_N$ with the sides $B_W = \{0\}\times[0,W]$ (west), $B_E = \{L\}\times[0,W]$ (east), $B_S = [0,L]\times\{0\}$ (south) and $B_N = [0,L]\times\{W\}$ (north). The object consists of a solid material with density $\rho > 0$, specific heat capacity $c > 0$ and anistropic thermal conductivity $\lambda_1 > 0$ along x_1-axis (or length) and $\lambda_2 > 0$ along x_2-axis (or width).

The anisotropy describes the physical situation to conduct heat faster along one axis compared to the other axis. We summarize these material properties as diffusivity constants $\alpha_1 := \frac{\lambda_1}{c\rho}$ and $\alpha_2 := \frac{\lambda_2}{c\rho}$.

The evolution of temperature in the plate $\theta : [0,T] \times \Omega \to \mathbb{R}$ solves the two-dimensional heat equation

$$\frac{d}{dt}\theta(t,x) = \alpha_1 \frac{\partial^2}{\partial x_1^2}\theta(t,x) + \alpha_2 \frac{\partial^2}{\partial x_2^2}\theta(t,x) \tag{1}$$

for $(t,x) \in (0,T_{final}] \times \Omega$, with an initial temperature distribution $\theta(0,x) = \theta_0(x)$ and boundary conditions

$$\lambda_1 \frac{\partial}{\partial x_1}\theta(t,x) = 0 \, , x \in B_W \cup B_E, \tag{2}$$

$$\lambda_2 \frac{\partial}{\partial x_2}\theta(t,x) = \begin{cases} -\phi_{in}(t,x) & \text{for } x \in B_S, \\ 0 & \text{for } x \in B_N. \end{cases} \tag{3}$$

The boundary conditions in Eq. (2, 3) describe a thermal insulation (no thermal losses) and heat can only be supplied along boundary B_S via the heat flux

$$\phi_{in}(t,x) = \sum_{n=1}^{N_u} b_n(x)\; u_n(t) \tag{4}$$

with $N_u \geq 1$ as the number of actuators. We assume that the boundary side B_S consists of segments $\beta_n \subseteq B_S$ such that $B_S = \bigcup_{n=1}^{N_u} \beta_n$ and each actuator operates only on its segment and has the spatial characteristics

$$b_n(x) = \begin{cases} m_n \exp\left(\|M_n [x - x_{c,n}]\|^{2\nu_n}\right) & \text{for } x \in \beta_n \\ 0 & \text{for } x \in B_S \setminus \beta_n \end{cases}$$

with $m_n \in [0,1]$, $M_n > 0$ and $\nu_n \in \mathbb{N}_{>0}$ for $n \in \{1,\dots,N_u\}$. Furthermore, we assume an arbitrary positive input signal $u_n : [0,T) \to \mathbb{R}_{>0}$. Analog to the actuator setup, we consider temperature measurements on boundary side B_N. Each temperature sensor operates only on its segment $\gamma_n \subseteq B_N$ with $B_N = \bigcup_{n=1}^{N_y} \gamma_n$ where $N_y \geq 1$ denotes the number of sensors. We assume the temperature measurement

$$y_n(t) = \frac{1}{\int_{\gamma_n} g_n(x)dx} \int_{\gamma_n} g_n(x)\; \theta(t,x)dx \tag{5}$$

for $n \in \{1,\dots,N_y\}$ and with the sensor characterization $g_n : \gamma_n \to [0,1]$ analog to b_n. We approximate the spatially derivatives in heat equation (1) next, to derive the large-scale state space system. The heat conduction modeling approach with the actuator and sensor characteristics is also described in [3, 4].

We assume that all finite difference nodes are inside the rectangle as $x_{j,m} := ([j - \frac{1}{2}]\Delta x_1, [m - \frac{1}{2}]\Delta x_2)^\top$ for $j \in \{1,2,\dots,J\}$ and $m \in \{1,2,\dots,M\}$ with $J > 0$ and $M > 0$, see the dots in Fig. 1. We introduce the global index $i(j,k) := j + (k-1)\;J$ and the temperature vector of the nodes $\Theta(t) := (\theta(t,x_1),\dots,\theta(t,x_i),\theta(t,x_{N_c}))$ with totals number of nodes $N_c = J\;M$. We approximate the second-order derivatives with finite differences $\frac{\partial^2}{\partial x_1^2} \approx \frac{1}{\Delta x_1^2} D_1$ in x_1-direction with $D_1 = \operatorname{diag}(\underbrace{\tilde{D}_1,\dots,\tilde{D}_1}_{M \text{ matrix blocks}})$ and

$$\tilde{D}_1 = \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix} \in \mathbb{R}^{J \times J}$$

and for the x_2-direction we find $\frac{\partial^2}{\partial x_2^2} \approx \frac{1}{\Delta x_2^2} D_2$ with

$$D_2 = \begin{pmatrix} -I_J & I_J & & & \\ I_J & -2I_J & I_J & & \\ & \ddots & \ddots & \ddots & \\ & & I_J & -2I_J & I_J \\ & & & I_J & -I_J \end{pmatrix}.$$

Summing up D_1 and D_2 with its coefficients, we note the system matrix as

$$A = \frac{\alpha_1}{\Delta x_1^2} D_1 + \frac{\alpha_2}{\Delta x_2^2} D_2. \tag{6}$$

In the matrices $\tilde{D}_1$ and D_2, we already consider the thermally insulated boundary conditions but we have to include also the heat supply via

$$B = \frac{\alpha_2}{\lambda_2\;\Delta x_2}(\tilde{b}_1,\dots,\tilde{b}_{N_u})$$

with vectors $\tilde{b}_n := \left(b_n(x_{1,1}),\dots,b_n(x_{J,1}),0_{J(M-1)}\right)^\top$. So, we formulate the state space formulation

$$\frac{d}{dt}\Theta(t) = A\;\Theta(t) + B\;u(t). \tag{7}$$

with the temperature measurement $y(t) = C\;\Theta(t)$ from Eq. (5).

The output matrix is noted as $C = (\tilde{c}_1, \ldots, \tilde{c}_{N_y})^\top$ with $\tilde{c}_n := \left(0_{J\times(M-1)}, \overline{g}_1, \ldots, \overline{g}_{N_y}\right)$ and elements $\overline{g}_{n,j} = g_n(x_{j,M})/\sum_{j=1}^{J} g_n(x_{j,M})$. We know that we can find the analytical solution of Eq. (7) as

$$\Theta(t) = \mathrm{e}^{A[t-t_0]}\,\Theta(0) + \int_{t_0}^{t} \mathrm{e}^{A[t-\tau]}\,Bu(\tau)d\tau. \quad (8)$$

However, as the the size of matrix A grows quadratically by N_c, the exact and fast computation of $\exp(A)$ might be a problem for large systems $N_c \gg 1$. If we know the eigenvalues μ_i and eigenvectors ψ_i of A for $i \in \{1,\ldots,N_c\}$, then we find the matrix exponential as

$$\exp(A\ t) = V^{-1}\,\mathrm{diag}(\mathrm{e}^{\mu_1\cdot \mathrm{t}}, \ldots, \mathrm{e}^{\mu_{\mathrm{JM}}\cdot \mathrm{t}})\,V \quad (9)$$

with $V = [\psi_1, \ldots, \psi_{JM}]$. The article [5] states explicit formulas for the pair (μ_i, ψ_i) for a 1-dim. heat equation problem and we extend this concept for our 2-dim. problem. We assume $f(z,n) := \cos(z(2n-1)\pi)$ and we state the eigenvalues of the 2-dim. problem as

$$\begin{aligned}\mu_{j,m} = &-2p_1\left[1 - f([j-1]/J, 1)\right]\\ &-2p_2\left[1 - f([m-1]/M, 1)\right]\end{aligned} \quad (10)$$

with $(j,m) \in \{1,\ldots,J\} \times \{1,\ldots,M\}$, $p_l = \alpha_l/\Delta x_l^2$ and $l \in \{1,2\}$. We note the eigenvectors

$$\psi_i = (\psi_{i,1}, \ldots \psi_{i,JM})^\top$$

with vector elements

$$\psi_{(j,m),(\tilde{n}_j,\tilde{n}_m)} = f\left(\frac{j-1}{2J}, \tilde{n}_j\right)\, f\left(\frac{m-1}{2M}, \tilde{n}_m\right) \quad (11)$$

for the indices $(\tilde{n}_j, \tilde{n}_m) \in \{1,\ldots,J\} \times \{1,\ldots,M\}$. The proof of this assumption is omitted here, the correctness of (μ_i, ψ_i) can be verified by evaluating $A\psi = \mu\psi$ with cosine identities.

2 Controller Design

We sample the temperature in time as $\Theta(n\Delta T) =: \Theta(n)$ for $n \in \{0, \ldots, \lfloor \frac{T}{\Delta T} \rfloor\}$ with sampling time $\Delta T > 0$ and we derive from Eqs. (8-11) the time-sampled system

$$\Theta(n+1) = A_D\,\Theta(n) + B_D\,u(n) \quad (12)$$

with matrices

$$A_D := \exp(A\Delta T) = V^{-1}\,\mathrm{diag}(\mathrm{e}^{\mu_1\cdot\Delta T}, \ldots, \mathrm{e}^{\mu_{JK}\cdot\Delta T})V,$$
$$B_D := \int_0^{\Delta T} \exp\left(A\left[\Delta T - \tau\right]\right)Bd\tau.$$

The input signal $u(n) := -K\ \Theta(n) + W\ r(n)$ is designed such that a state feedback $K \in \mathbb{R}^{N_u \times J\ K}$ stabilizes the system and a static feed-forward filter $W \in \mathbb{R}^{N_u \times N_y}$ steers the measured temperatures to a static reference signal $r \in \mathbb{R}^{N_y}$. Feedback matrix K is found by solving the discrete infinite-horizon linear-quadratic regulator problem

$$\min_u \sum_{n=1}^{\infty} \Theta(n)^\top Q\,\Theta(n) + u(n)^\top R\,u(n) \quad (13)$$

with subject to Eq. (12). The optimal control problem (13) leads to the discrete time algebraic Riccati equation

$$P = Q + A_D^\top P A_D - \left[A_D^\top P B_D\right]\left[R + B_D^\top P B_D\right]^{-1}\left[B_D^\top P A_D\right]$$

where we compute matrix P to obtain the feedback gain

$$K = \left[R + B_D^\top P B_D\right]^{-1} B_D^\top P A_D.$$

If the number of actuators and sensors coincide, $N_u \equiv N_y$, then we find the filter matrix W to drive the thermal dynamics to a constant reference value r. We assume for $n \to \infty$ an uniform temperature distribution

$$\left[A_D - B_D\ K\right]\,\Theta(n) + B_D W r = \Theta(n) \quad (14)$$

and a constant output

$$y(n) = C\Theta(n) = r. \quad (15)$$

We identify Θ in Eq. (15) with Eq. (14) and we use the reference tracking $y \equiv r$ to formulate the filter

$$W = -\left[C(A_d - B_d K - I)^{-1} B_d\right]^{-1}.$$

For more details on the control design, we refer to introductory text books, e.g. [6, Ch. 7].

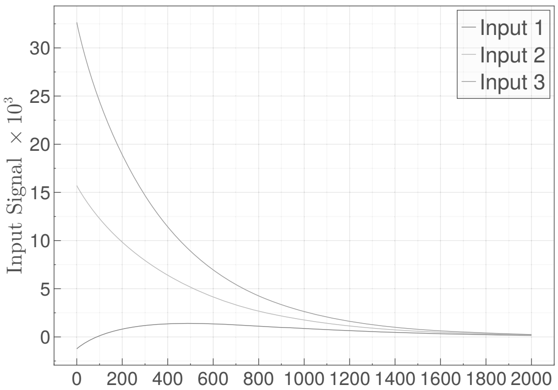

Figure 2: Input signals u_n with $n \in \{1,2,3\}$ applied on boundary side B_S, time t in [s].

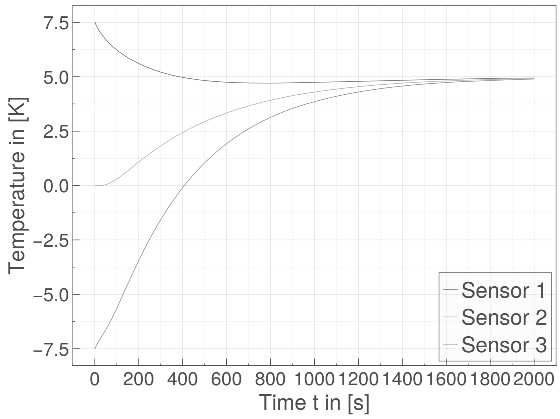

Figure 3: Measured temperatures y_n with $n \in \{1,2,3\}$ on boundary side B_N.

3 Simulation Case Study

We apply the proposed concepts on an example of a steel plate with model parameters as in Table 1 and controller configuration as in Table 2.

The initial temperature distribution is $\theta_0(x_1,x_2) = 10\sin(2\pi\frac{x_1}{L})$. We design the controller such that it is forced to act quickly $Q \gg R$ and the static filter shall steer the three measured temperatures towards the reference value $r = (5,5,5)^\top$.

The computed input signal $u(t)$ and the resulting measurement temperatures $y(t)$ are depicted in Figure 2 and Figure 3. The input signals u_2 and u_3 start with high initial values compared to u_1 because they have to increase the temperatures of y_2 and y_3, see Figure 3. All temperatures along boundary side B_N converge in Figure 4 after ca. 1200 seconds. We implemented the simulation with JULIA programming language [7] and solved the algebraic Riccati equation with the library MATRIXEQUATIONS.JL [8]. The full source code is available [9].

L	W	J	K	(λ_1,λ_2)	ρ	c
0.3	0.1	30	10	$(40,60)$	8000	400

Table 1: System Parameters for Simulation.

Actuators		Sensors				
N_u	(m,M,ν)	N_y	(m,M,ν)	ΔT	Q	R
3	$(1,40,2)$	3	$(1,20,1)$	2	$10^5 I$	I

Table 2: Controller Parameters for Simulation.

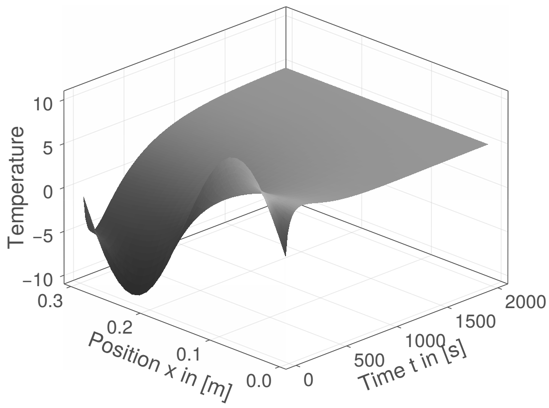

Figure 4: Temperature evolution on boundary side B_N.

Discussion & Conclusion

In real world applications, we have to deal with thermal emissions like convection and radiation on each boundary side, which we did not consider in this contribution. Moreover, in 3-dim. objects we can only measure the temperatures on the boundary, not inside the object. However, as we require access to all temperatures for the full state feedback, we have to compute them with a state observer. In a nutshell, we derived a state space control design for a 2-dim. heat equation and we exemplified its performance. Further research will focus on the extension of our approach for systems with thermal emissions and 3-dim. geometries.

References

[1] Tay A, Tan KK, Zhao S, Lee TH. Predictive Ratio Control of Multizone Thermal Processing System in Lithography, *IFAC Proceedings Volumes*. 2008; 41(2). 10863–10868.

[2] Meurer T. Control of higher–dimensional PDEs: Flatness and backstepping designs. Springer Science & Business Media, 2012.

[3] Scholz S, Berger L. Hestia. jl: A Julia Library for Heat Conduction Modeling with Boundary Actuation. *Simul. Notes Eur.*. 2023; 33(1): 27–30, DOI 10.11128/sne.33.sn.10634.

[4] Scholz S, Berger L. Optimization-based Reference Tracking for Two-Dimensional Multiple Source Heating, *2024 IEEE 63rd Conference on Decision and Control*. 2024. DOI 10.1109/CDC56724.2024.10886335

[5] Yueh WC, Eigenvalues of several tridiagonal matrices. *Applied Mathematics E-Notes [electronic only]*. 2005; 5: 66–74.

[6] Åström KJ, Murray R. Feedback systems: an introduction for scientists and engineers. Princeton university press, 2021.

[7] Bezanson J, Edelman A, Karpinski S, Shah VB. Julia: A fresh approach to numerical computing. *SIAM Review*. 2017; 59(1): 65-98.

[8] Varga A, Karrasch D, Legat B, Aluthge D. andreasvarga/MatrixEquations.jl: v2.4.2 (v2.4.2). Zenodo. 2024. Available: DOI 10.5281/zenodo.10546715

[9] Scholz S. stephans3/StateSpace2DHeatConduction.jl: StateSpace2DHeatConduction.jl v0.1.0 (v0.1.0). Zenodo. 2024. Available: DOI 10. 5281/zenodo.11350656

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